BASIC MATHS & LOGARITHM

NUMBER SYSTEM

Natural Numbers : Counting numbers 1, 2, 3, 4, 5,..... are known as natural numbers. The set of all natural numbers can be represent by N = {1, 2, 3, 4, 5,....}

Whole Numbers : If we include 0 among the natural numbers, then the numbers 0, 1, 2, 3, 4, 5..... are called whole numbers. The set of whole numbers can be represented by W = {0, 1, 2, 3, 4, 5,....}Clearly, every natural number is a whole number but 0 is a whole number which is not a natural number.

Integers : All counting numbers and their negatives including zero are known as integers. The set of integers can be represented by Z = {.... –4, –3, –2, –1, 0, 1, 2, 3, 4,....}

Remarks

(i) Positive integers I^+ = {1, 2, 3 ....} = N
(ii) Negative integers I^- = {...., –3, –2, –1}.
(iii) Non-negative integers (whole numbers) = {0, 1, 2, ...}.
(iv) Non-positive integers = {...., –3, –2, –1, 0}.

Even and Odd Numbers

Number which are divisible by 2 are called even numbers.
Numbers which are not divisible by 2 are called odd number.
In general, even numbers can be represented by $2n$ and odd numbers can be represented by $2n \pm 1$.
(where n is an integer)

(i) The sum and product of any number of even numbers is an even number.
(ii) The difference of two even numbers is an even number.
(iii) The sum of odd numbers depends on the number of numbers.
(iv) If the number of numbers is odd, then sum is an odd number.
(v) If the number of numbers is even, then sum is an even number.
(vi) If the product of a certain number is even, then atleast one of the number has to be even.

Prime Numbers : Natural numbers which are divisible by 1 and itself only are called prime numbers. Let 'p' be a natural number, 'p' is said to be prime if it has exactly two distinct positive integral factors, namely 1 and itself. e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 .....

Composite Numbers : Let ‘a’ be a natural number ‘a’ then it is said to be composite if it has atleast 3 distinct factors, that means ‘a’ has more than two divisors. eg. 4, 6, 8, 9, 10

(i) '1' is neither prime nor composite.
(ii) '2' is the only even prime number.
(iii) '4' is the smallest composite number.
(iv) Natural numbers which are not prime are composite numbers (except 1)

Co-Prime Numbers : Two natural numbers (not necessarily prime) are called coprime, if there H.C.F (Highest common factor) is one. e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (5, 6), (7, 8) (15, 16) etc. These numbers are also called as **relatively prime** numbers.

Twin Prime Numbers : If the difference between two prime numbers is two, then the numbers are called twin prime numbers. e.g. {3, 5}, {5, 7}, {11, 13}, {17, 19}, {29, 31}

(i) 1 is neither a prime number nor a composite number.

(ii) Numbers which are not prime are composite numbers (except 1).

(iii) '4' is the smallest composite number.

(iv) '2' is the only even prime number.

(v) Two prime number(s) are always co-prime but converse need not be true.

(vi) Consecutive natural numbers are always co-prime numbers.

Rational Numbers (Q) : All the numbers that can be represented in the form p/q, where p and q are integers and q ≠ 0, are called rational numbers. Integers, Fractions, Terminating decimal numbers, Non-terminating but repeating decimal numbers are all rational numbers.

$$Q = \left\{\frac{p}{q} : p, q \in I \text{ and } q \neq 0\right\}$$

Irrational Numbers (Q^C) : There are real numbers which can not be expressed in p/q form. Non-Terminating non repeating decimal numbers are irrational number

e.g. $\sqrt{2}, \sqrt{5}, \sqrt{3}, \sqrt[3]{10}$; e, π. $e \approx 2.71$ is called Napier's constant and $\pi \approx 3.14$

(i) Integers are rational numbers, but converse need not be true.

(ii) A rational number always exists between two distinct rational numbers, hence infinite rational numbers exist between two rational numbers.

(iii) Sum of a rational number and an irrational number is an irrational number e.g. $2 + \sqrt{3}$

(iv) If $a \in Q$ and $b \notin Q$, then ab = rational number, only if a = 0.

(v) Sum, difference, product and quotient of two irrational numbers need not be an irrational number or we can say, result may be a rational number also.

Ex. Express the following rational numbers in the form of $\frac{p}{q}$, (where $p, q \in I$)

(i) $0.1\overline{2}$ **(ii)** $1.5\overline{23}$

Sol. **(i)** Let $x = 0.1\overline{2} = 0.1222.......$

$10x = 1.\overline{2}$ ....**(i)**

$100x = 12.\overline{2}$ ....**(ii)**

$\Rightarrow$ $90x = 11$ $\Rightarrow$ $x = \frac{11}{90}$ (so x is a rational number)

(ii) Let $x = 1.5\overline{23}$ $\Rightarrow$ $10x = 15.\overline{23}$

$1000x = 1523.\overline{23}$

$990x = 1508$ $\Rightarrow$ $x = \frac{1508}{990} = \frac{754}{495}$ (so x is a rational number)

Complex Numbers : A number of the form a + ib is called a complex number, where $a, b \in R$ and $i = \sqrt{-1}$. Complex number is usually denoted by Z and the set of complex number is represented by C. Thus $C = \{a + ib : a, b \in R \text{ and } i = \sqrt{-1}\}$

(i) $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ however in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

(iii) Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively

i.e. $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

LCM and HCF

(a) The highest common factor (H.C.F) of two (or more) numbers is the largest number that divides evenly into both numbers. In other words the H.C.F. is the largest of all the common factors.

The common factors of 12 and 18 are 1, 2, 3 and 6.

The largest common factor is 6. So this is the H.C.F. of 12 and 18.

(b) The Lowest Common Multiple (L.C.M) is the smallest number that is a common multiple of two or more numbers. The L.C.M of 3 and 5 is 15.

(c) The product of HCF and LCM of two numbers (or expressions) is equal to the product of the numbers.

(d) LCM of $\left(\frac{a}{b}, \frac{p}{q}, \frac{l}{m}\right) = \frac{L.C.M.\text{ of (a, p, l)}}{H.C.F.\text{ of (b, q, m)}}$

Divisibility Rules

Divisibility of	Test
2	The digit at the unit place of the number is divisible by 2.
3	The sum of digits of the number is divisible by 3.
4	The last two digits of the number together are divisible by 4.
5	The digit of the number at the unit place is either 0 or 5.
6	The digit at the unit place of the number is divisible by 2 & the sum of all digits of the number is divisible by 3.
8	The last 3 digits of the number all together are divisible by 8.
9	The sum of all it's digits is divisible by 9.
10	The digit at unit place is 0.
11	The difference between the sum of the digits at even places and the sum of digits at odd places is 0 or multiple of 11. e.g.1298, 1221, 123321, 12344321, 1234554321, 123456654321

Ex. Consider a number N = 2 1 P 5 3 Q 4

(i) Find number of ordered pairs (P, Q) so that the number 'N' is divisible by 9.

(ii) Find number of values of Q so that the number 'N' is divisible by 8.

(iii) Find number of ordered pairs (P, Q) so that the number 'N' is divisible by 44.

Sol. **(i)** Sum of digits = P + Q + 15

'N' is divisible by 9 if P + Q + 15 = 18, 27

$\Rightarrow$ P + Q = 3 .....**(i)**

or P + Q = 12 .....**(ii)**

P = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

From equation **(i)**

$$\left.\begin{array}{ll} P=0, & Q=3 \\ P=1, & Q=2 \\ P=2, & Q=1 \\ P=3, & Q=0 \end{array}\right\}$$ No. of ordered pairs is 4   From equation **(ii)**

$$\left.\begin{array}{ll} P=3, & Q=9 \\ P=4, & Q=8 \\ \ldots\ldots & \ldots\ldots \\ P=8, & Q=4 \\ P=9, & Q=3 \end{array}\right\}$$ No. of ordered pairs is 7

Total number of ordered pairs is 11

(ii) N is divisible by 8 if

Q = 0, 4, 8

Number of values of Q is 3

(iii) $S_O = P + 9$

$S_E = Q + 6$

$S_E - S_0 = Q - P - 3$

N is divisible by 4 if

Q = 0, 2, 4, 6, 8

'N' is divisible is 11 if

Q – P – 3 = 0 or multiple of 11

P – Q = –3 ....**(i)**

or P – Q = 8 ....**(ii)**

From Equation (i)			**From Equation (ii)**		
Q = 0,	P = –3	**(not possible)**	Q = 0,	P = 8	
Q = 2,	P = –1	**(not possible)**	Q = 2,	P = 10	**(not possible)**
Q = 4,	P = 1		Q = 4,	P = 12	**(not possible)**
Q = 6,	P = 3		Q = 6,	P = 14	**(not possible)**
Q = 8,	P = 5		Q = 8,	P = 16	**(not possible)**

$\therefore$ number of ordered pairs is 3 $\therefore$ number of ordered pairs is 1

$\therefore$ total number of ordered pairs, so that number 'N' is divisible by 44 is 4

Some Important Identities

(1) $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$

(2) $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$

(3) $a^2 - b^2 = (a + b)(a - b)$

(4) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(5) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(6) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$

(7) $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$

(8) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

(9) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(10) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$$= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

(11) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$

(12) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

Ex. Show that the expression, $(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz).(y^2 - zx).(z^2 - xy)$ is a perfect square and find its square root.

Sol. $(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - zx)(z^2 - xy)$

$= a^3 + b^3 + c^3 - 3abc$ where $a = x^2 - yz$, $b = y^2 - zx$, $c = z^2 - xy$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2)$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)[(x^2 - yz - y^2 + zx)^2 + (y^2 - zx - z^2 + xy)^2 + (z^2 - xy - x^2 + yz)^2]$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)[\{x^2 - y^2 + z(x - y)\}^2 + \{y^2 - z^2 + x(y - z)\}^2 + \{z^2 - x^2 + y(z - x)\}^2]$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z)^2[(x - y)^2 + (y - z)^2 + (z - x)^2]$

$= (x + y + z)^2(x^2 + y^2 + z^2 - xy - yz - zx)^2 = (x^3 + y^3 + z^3 - 3xyz)^2$ **(which is a perfect square)**

its square roots are

$\pm(x^3 + y^3 + z^3 - 3xyz)$

Remainder Theorem : If a polynomial $a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_n$ is divided by $x - p$, then the remainder is obtained by putting $x = p$ in the polynomial.

Factor Theorem : A polynomial $a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_n$ is divisible by $x - p$, if the remainder is zero i.e. if $a_1p^n + a_2p^{n-1} + \ldots + a_n = 0$ then $x - p$ will be a factor of polynomial.

Ex. Show that $(x - 3)$ is a factor of the polynomial $x^3 - 3x^2 + 4x - 12$.

Sol. Let $p(x) = x^3 - 3x^2 + 4x - 12$ be the given polynomial. By factor theorem, $(x - a)$ is a factor of a polynomial $p(x)$ iff $p(a) = 0$. Therefore, in order to prove that $x - 3$ is a factor of $p(x)$, it is sufficient to show that $p(3) = 0$. Now,

$$p(x) = x^3 - 3x^2 + 4x - 12$$

$$\Rightarrow \quad p(3) = 3^3 - 3 \times 3^2 + 4 \times 3 - 12$$

$$= 27 - 27 + 12 - 12 = 0$$

Hence, $(x - 3)$ is a factor of $p(x) = x^3 - 3x^2 + 4x - 12$.

Ex. The polynomials $P(x) = kx^3 + 3x^2 - 3$ and $Q(x) = 2x^3 - 5x + k$, when divided by $(x - 4)$ leave the same remainder. Find the value of k.

Sol. $P(4) = 64k + 48 - 3 = 64k + 45$

$Q(4) = 128 - 20 + k = k + 108$

given $P(4) = Q(4)$

$\therefore \quad 64k + 45 = k + 108$

$\Rightarrow \quad 63k = 63 \quad \Rightarrow \quad k = 1$

Ratio and Proportion

(a) If $\frac{a}{b} = \frac{c}{d}$, then : $\frac{a+b}{b} = \frac{c+d}{d}$ (componendo); $\frac{a-b}{b} = \frac{c-d}{d}$ (dividendo);

$\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (componendo and dividendo); $\frac{a}{c} = \frac{b}{d}$ (alternendo); $\frac{b}{a} = \frac{d}{c}$ (invertendo)

(b) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$, then each ratio $= \left(\frac{a^n + c^n + e^n}{b^n + d^n + f^n}\right)^{\frac{1}{n}}$

Ex. If $x : y = 3 : 4$, then find the ratio of $7x - 4y : 3x + y$

Sol. $\frac{x}{y} = \frac{3}{4} \quad \therefore \quad 4x = 3y \quad$ or $\quad x = \frac{3}{4}y$

Now $\quad \frac{7x - 4y}{3x + y} = \frac{7 . \frac{3}{4}y - 4y}{3 . \frac{3}{4}y + y}$ (putting the value of x)

$$= \frac{\frac{21}{4}y - 4y}{\frac{9}{4}y + y} = \frac{5y}{13y} = \frac{5}{13} \qquad \text{i.e.} \qquad 5 : 13$$

Ex. If a, b, c, d, e are in continued proportion, then prove that

$$(ab + bc + cd + de)^2 = (a^2 + b^2 + c^2 + d^2)\ (b^2 + c^2 + d^2 + e^2)$$

Sol. If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}$, then we have

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \frac{\sqrt{(a^2 + b^2 + c^2 + d^2)}}{\sqrt{(b^2 + c^2 + d^2 + e^2)}} = k \text{ (say)}$$

i.e. $a = bk \quad \therefore \quad ab = b^2k$

$b = ck \quad \therefore \quad bc = c^2k$

$c = dk \quad \therefore \quad cd = d^2k$

$d = ek \quad \therefore \quad de = e^2k$

Again $(a^2 + b^2 + c^2 + d^2) = k^2 (b^2 + c^2 + d^2 + e^2)$ ....**(i)**

Now L.H.S. $= (ab + bc + cd + de)^2$

$= (kb^2 + kc^2 + kd^2 + ke^2)^2$

$= k^2 (b^2 + c^2 + d^2 + e^2)^2$

$= k^2 (b^2 + c^2 + d^2 + e^2) (b^2 + c^2 + d^2 + e^2)$

$= (a^2 + b^2 + c^2 + d^2) (b^2 + c^2 + d^2 + e^2)$ (Note) (use **(i)**)

Hence $(ab + bc + cd + de)^2 = (a^2 + b^2 + c^2 + d^2) (b^2 + c^2 + d^2 + e^2)$

Ex. Simplify $a\left(\dfrac{\sqrt{a}+\sqrt{b}}{2b\sqrt{a}}\right)^{-1} + b\left(\dfrac{\sqrt{a}+\sqrt{b}}{2a\sqrt{b}}\right)^{-1}$

Sol. The given expression is equal to

$$a\left(\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)+b\left(\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right) = 2ab\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}+\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right) = 2ab$$

Ex. Evaluate $\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{7+\sqrt{48}}}}$

Sol. $\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{7+\sqrt{48}}}} = \sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{4+3+2\sqrt{12}}}}$

$$= \sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{4}+\sqrt{3}}}$$

$$= \sqrt{3+\sqrt{3}+\sqrt{4+2\sqrt{3}}} = \sqrt{3+\sqrt{3}+\sqrt{3}+1} = \sqrt{4+2\sqrt{3}} = \sqrt{3}+1$$

Surds : If a is a positive rational number, which is not the nth power (n is any natural number) of any rational number, then the irrational number $\pm\sqrt[n]{a}$ are called simple surds or monomial surds. Every surd is an irrational number (but every irrational number is not a surd). So, the representation of monomial surd on a number line is same that of irrational numbers. Eg.

(a) $\sqrt{3}$ is a surd and $\sqrt{3}$ is an irrational number **(b)** $\sqrt[3]{5}$ is surd and $\sqrt[3]{5}$ is an irrational number

(c) π is an irrational number, but it is not a surd.

(d) $\sqrt[3]{3+\sqrt{2}}$ is an irrational number. It is not a surd, because $3+\sqrt{2}$ is not a rational number.

Laws of Surds

(i) $\left(\sqrt[n]{a}\right)^n = \sqrt[n]{a^n} = a$

(ii) $\sqrt[n]{a}\times\sqrt[n]{b} = \sqrt[n]{ab}$ [Here order should be same]

(iii) $\sqrt[n]{a}\div\sqrt[n]{b} = \sqrt[n]{\dfrac{a}{b}}$

(iv) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = \sqrt[m]{\sqrt[n]{a}}$ e.g., $\sqrt{\sqrt{\sqrt{2}}} = \sqrt[8]{2}$

(v) $\sqrt[n]{a} = \sqrt[n\times p]{a^p}$ or $\sqrt[n]{a^m} = \sqrt[n\times p]{a^{m\times p}}$ [Important for changing order of surds]

Rationalization of Surds : Process of converting surd into rational number is called it's rationalization. It is carried out by multiplying surd with an appropriates rationalizing factor.

(i) The conjugate surd of $\sqrt{a}+\sqrt{b}$ is $\pm(\sqrt{a}-\sqrt{b})$.

(ii) To rationalize $\frac{1}{\sqrt{a}+\sqrt{b}}$, multiply it by $\frac{(\sqrt{a}-\sqrt{b})}{(\sqrt{a}-\sqrt{b})}$ or $\frac{(\sqrt{b}-\sqrt{a})}{(\sqrt{b}-\sqrt{a})}$.

(iii) If $a+\sqrt{b}=c+\sqrt{d}$, then a = c and b = d.

(iv) To find $\sqrt{(a+\sqrt{b})}$ write it in the form $\sqrt{m+n+2\sqrt{mn}}$, such that m + n = a and 4 mn = b, then

$\sqrt{(a+\sqrt{b})}=\pm(\sqrt{m}+\sqrt{n})$

(v) $\sqrt{a\sqrt{a\sqrt{a....\infty}}}=a$

(vi) $\sqrt{a\sqrt{a\sqrt{a\sqrt{a......n\ \text{times}}}}}=a^{1-\frac{1}{2^n}}$

(vii) If $\sqrt{a+\sqrt{a+\sqrt{a+...\infty}}}=p$, then $p(p-1)=a$.

(viii) The rationalizing factor of $a^{1/3}+b^{1/3}$ is
$a^{2/3}-(ab)^{1/3}+b^{2/3}$.

(ix) The rationalising factor of $a^{1/3}-b^{1/3}$ is
$a^{2/3}+(ab)^{1/3}+b^{2/3}$.

(x) The rationalising factor of $\sqrt{a}+\sqrt{b}$ is $\sqrt{a}-\sqrt{b}$.

(xi) The rationalising factor of $\sqrt{a}-\sqrt{b}$ is $\sqrt{a}+\sqrt{b}$.

(xii) The rationalising factor of $a+\sqrt{b}$ is $a-\sqrt{b}$.

Ex. Arrange, $\sqrt{2}$, $\sqrt[3]{3}$ and $\sqrt[4]{5}$ in ascending order.

Sol. $\sqrt{2}$, $\sqrt[3]{3}$ and $\sqrt[4]{5}$

L.C.M. of 2, 3, 4 is 12.

$\therefore$ $\sqrt{2}=\sqrt[2\times6]{2^6}=\sqrt[12]{64}$

$\sqrt[3]{3}=\sqrt[3\times4]{3^4}=\sqrt[12]{81}$

$\sqrt[4]{5}=\sqrt[4\times3]{5^3}=\sqrt[12]{125}$

As, $64<81<125$.

$\therefore$ $\sqrt[12]{64}<\sqrt[12]{81}<\sqrt[12]{125}$

$\Rightarrow$ $\sqrt{2}<\sqrt[3]{3}<\sqrt[4]{5}$

Ex. If $\dfrac{3+2\sqrt{2}}{3-\sqrt{2}} = a+b\sqrt{2}$, where a and b are rationals then, find the values of a and b.

Sol. L.H.S $\dfrac{3+2\sqrt{2}}{3-\sqrt{2}} = \dfrac{(3+2\sqrt{2})(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}$

$$= \frac{9+3\sqrt{2}+6\sqrt{2}+4}{9-2}$$

$$= \frac{13+9\sqrt{2}}{7}$$

$$= \frac{13}{7}+\frac{9}{7}\sqrt{2}$$

$$\therefore \quad \frac{13}{7}+\frac{9}{7}\sqrt{2} = a+b\sqrt{2}$$

Equating the rational and irrational parts

We get $a = \dfrac{13}{7}$, $b = \dfrac{9}{7}$

If 'a' is any non zero real or imaginary number and 'm' is the positive integer, then a^m = a. a. a. ...a (m times). Here a is called the base and m is called the index, power or exponent.

Ex. Prove that : $\dfrac{x^{-1}}{x^{-1}+y^{-1}} + \dfrac{x^{-1}}{x^{-1}-y^{-1}} = \dfrac{2y^2}{y^2-x^2}$

Sol. $\dfrac{x^{-1}}{x^{-1}+y^{-1}} + \dfrac{x^{-1}}{x^{-1}-y^{-1}}$

$$= \frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{y}} + \frac{\frac{1}{x}}{\frac{1}{x}-\frac{1}{y}} = \frac{\frac{1}{x}}{\frac{y+x}{xy}} + \frac{\frac{1}{x}}{\frac{y-x}{xy}}$$

$$= \frac{xy}{x(y+x)} + \frac{xy}{x(y-x)} = \frac{xy(y-x)+xy(y+x)}{x(y^2-x^2)} \text{ c}$$

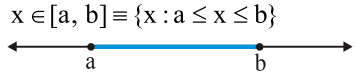
$$= \frac{y(y-x)+y(y+x)}{y^2-x^2} = \frac{y^2-xy+y^2+xy}{y^2-x^2} = \frac{2y^2}{y^2-x^2}$$

Intervals

The set of numbers any two real numbers is called interval. The following are the types of interval.

I Closed Interval

$x \in [a, b] \equiv \{x : a \le x \le b\}$

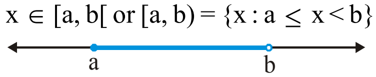


II Open Interval

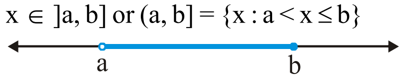
$x \in (a, b)$ or $]a, b[\equiv \{x : a < x < b\}$

III Semi Open or Semi-Closed Interval

$x \in [a, b[$ or $[a, b) = \{x : a \le x < b\}$

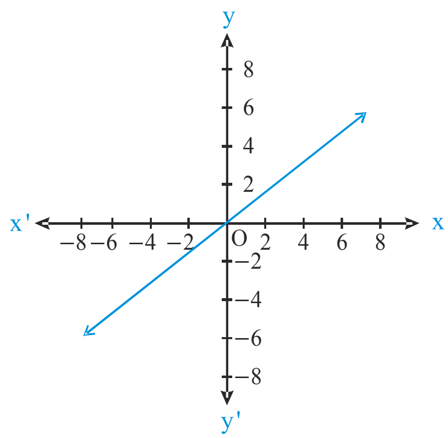


$x \in]a, b]$ or $(a, b] = \{x : a < x \le b\}$



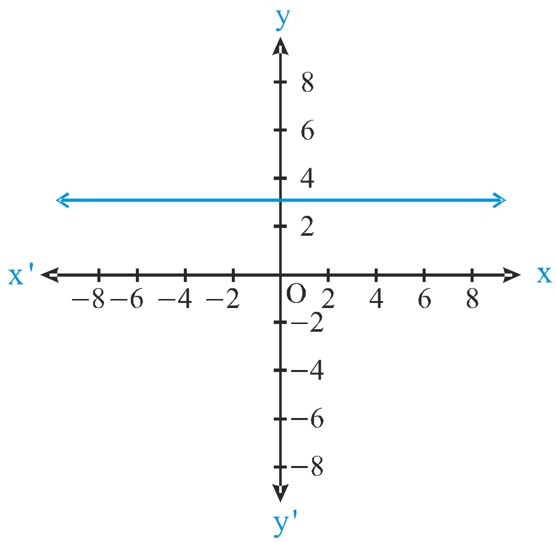
Some Function, their Graph and Property

(i) Identity Function : Let **R** be the set of real numbers. Define the real valued function $f : \mathbf{R} \to \mathbf{R}$ by $y = f(x) = x$ for each $x \in \mathbf{R}$. Such a function is called the identity function.



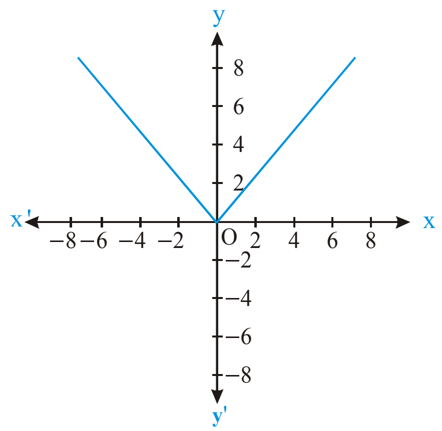
f(x) = x

(ii) Constant Function : Define the function f: $\mathbf{R} \to \mathbf{R}$ by $y = f(x) = c, x \in \mathbf{R}$ where c is a constant and each $x \in \mathbf{R}$. Here domain of f is **R** and its range is $\{c\}$. The graph is a line parallel to x-axis.



f(x) = 3

(iii) The Modulus Function : The function f: $\mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x|$ for each $x \in \mathbf{R}$ is called modulus function. For each non-negative value of x, $f(x)$ is equal to x. But for negative values of x, the value of $f(x)$ is the negative of the value of x, i.e.,



f(x) = |x|

Properties of Modulus Functions

(A) If a, b are positive real numbers, then

$x^2 \le a^2 \Leftrightarrow |x| \le a \Leftrightarrow -a \le x \le a$

$x^2 \ge a^2 \Leftrightarrow |x| \ge a \Leftrightarrow x \le -a$ or, $x \ge a$

$x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$

$x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a$ or, $x > a$

$a^2 \le x^2 \le b^2 \Leftrightarrow a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

$a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \quad \Leftrightarrow \in (-b, -a) \cup (a, b)$

(B) For real numbers x and y, we have

$|x + y| = |x| + |y|$, if $(x \ge 0$ and $y \ge 0)$ or, $(x, < 0$ and $y < 0)$

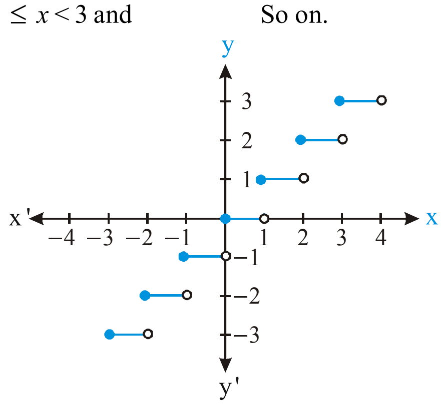
$|x - y| = |x| - |y|$, if $(x \ge 0$ and $|x| \ge |y|)$ or, $(x \le 0, y \le 0$ and $|x| \ge |y|)$

$|x \pm y| \le |x| + |y| \qquad |x \pm y| > \big||x| - |y|\big|$

(iv) **Greatest Integer Function :** The function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = [x], x \in \mathbf{R}$ assumes the value of the greatest integer, less than or equal to x. Such a function is called the **greatest integer function.** From the definition of $[x]$, we can see that

$[x] = -1$ for $-1 \le x < 0$ $\qquad [x] = 0$ for $0 \le x < 1$ $\qquad [x] = 1$ for $1 \le x < 2$

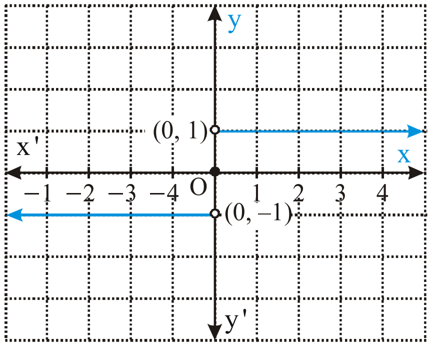
$[x] = 2$ for $2 \le x < 3$ and $\qquad$ So on.



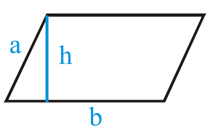
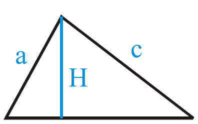
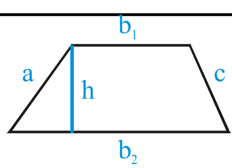
(v) **Signum Function :** The function of f defined by $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ or, $f(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$ is called the signum function.

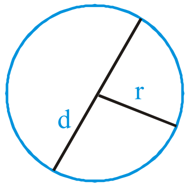
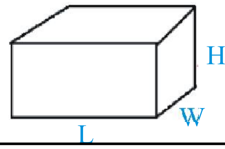
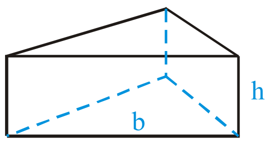
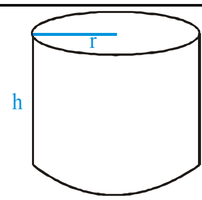
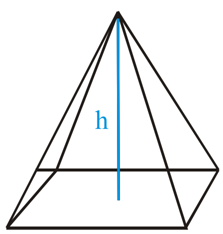
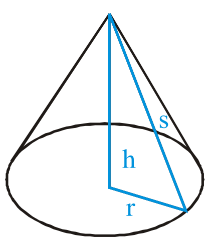
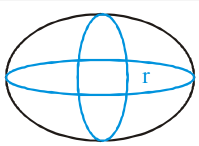
The domain of the signum function is the set R of all real numbers and the range is the set {–1, 0, 1}

The graph of the signum function is as shown in Fig.

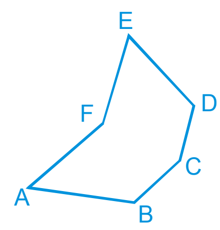


Formulas for Perimeter, Area, Surface, Volume

Shapes	Formulas
	**Rectangle** Area = Length × Width $A = LW$ **Perimeter** = 2 × Lengths + 2 × Widths $P = 2L + 2W$
	**Parallelogram** **Area** = Base X Height $A = bh$ **Perimeter** = add the length of all sides $P = 2a + 2b$
	**Triangle** **Area** = 1/2 of the base × the height Heron"s Formula for Area = $\sqrt{(s(s-a)(s-b)(s-c))}$ $s = (a + b + c) / 2$ **Perimeter** = $a + b + c$ (add the length of the three sides)
	**Trapezoid** **Area** = 1/2 of the base × the height $A=\left(\dfrac{b_1+b_2}{2}\right)h$ **Perimeter** = add lengths of all sides $P = a + b_1 + b_2 + c$

	**Circle** **Radius** = the distance from the center to a point on the circle (r). **Diameter** = the distance between two points on the circle through the center (d = 2r). Circumference = the distance around the circle ($C = \pi d = 2\pi r$). (Assume $\pi \approx 3.14$) **Area** = πr^2
	**Rectangular Solid** **Volume** = Length × Width × Height V = LWH **Surface** = 2LW + 2LH + 2WH
	**Prisms** **Volume** = Base × Height V = bh **Surface** = 2b + Ph (b is the area of the base P is the perimeter of the base)
	**Cylinder** **Volume** = πr^2 × height $V = \pi r^2 h$ **Surface** = 2π radius × height $S = 2\pi rh + 2\pi r^2$
	**Pyramid** **Volume** = 1/3 area of the base × height $V = \frac{1}{3}bh$ b is the area of the base **Surface Area :** Add the area of the base to the sum of the areas of all of the triangular faces. The areas of the triangular faces will have different formulas for different shaped bases.
	**Cones** **Volume** = 1/3 area of the base × height $V = \frac{1}{3}\pi r^2 h$ **Surface** $S = \pi r^2 + \pi rs$ $= \pi r^2 + \pi r\sqrt{r^2 + h^2}$
	**Sphere Volume** $V = \frac{4}{3}\pi r^3$ **Surface** $S = 4\pi r^2$

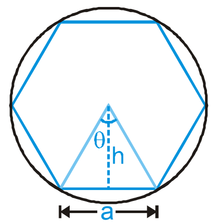
Polygon



A plane figure enclosed by line segments (sides of polygon).

(a) **n sides polygon have n sides :** Triangle and quadrilaterals are polygon of three and four sides respectively. The polygons having 5 to 10 sides are called, PENTAGON, HEXAGON, HEPTAGON, OCTAGON, NANOGON and DECAGON respectively.

(b) **Regular polygon :** Polygon which has all equal sides and equal angles and can be inscribed in a circle whose center coincides with the center of polygon. Therefore the center is equidistant from all its vertices.



(i) A regular polygon can also circumscribe a circle.

(ii) A 'n' sided regular polygon can be divided into 'n' Isosceles Congruent Triangles with a common vertex i.e. centre of polygon.

(iii) Area $= n \times \left(\frac{1}{2}\right) \times a \times h$

(iv) Perimeter = na

(v) Each interior angle of polygon $= \left(\frac{n-2}{n}\right) \times 180°$

(vi) Angle subtended at the centre of inscribed/circumscribed circle by one side = 360°/n

(vii) Each exterior angle $= \left(\frac{360}{n}\right)^{\circ}$

(viii) Sum of all interior angle = $(n-2) \times 180°$

(ix) Sum of all exterior angles = 360°

(x) **Convex polygon :** If any two consecutive vertices are joined then remaining all other vertices will lie on same side.

Some Facts about Inequalities

The following are some very useful points to remember.

1. $a \le b \Rightarrow$ either $a < b$ or $a = b$

2. $a < b$ and $b < c \Rightarrow a < c$

3. $a < b \Rightarrow -a > -b$, i.e., the inequality sign reverses if both sides are multiplied by a negative number

4. $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c$

5. $a < b \Rightarrow ka < kb$ if $k > 0$, and $ka > kb$ if $k < 0$

6. $0 < a < b \Rightarrow a^r < b^r$ of $r > 0$, and $a^r > b^r$ if $r < 0$

7. $a + \frac{1}{a} \ge 2$ for $a > 0$ and equality holds for $a = 1$

8. $a + \frac{1}{a} \le -2$ for $a < 0$ and eqaulity holds for $a = -1$

9. If $a > 2 \Rightarrow 0 < \frac{1}{x} < \frac{1}{2}$

10. If $x < -3 \Rightarrow -\frac{1}{3} < \frac{1}{x} < 0$

11. If $x < 2$, then we must consider $-\infty < x < 0$ or $0 < x < 2$

12. Squaring an inequality :

If $a < b$, then $a^2 < b^2$ does not follow always

Consider the following illustrations :

(a) $2 < 3 \Rightarrow 4 < 9$, but $-4 < 3 \Rightarrow 16 > 9$

(b) Also, $x > 2 \Rightarrow x^2 > 4$, but $x < 2 \Rightarrow x^2 \geq 0$

(c) $2 < x < 4 \Rightarrow 4 < x^2 < 16$

(d) $-2 < x < 4 \Rightarrow 0 \leq x^2 < 16$

(e) $-5 < x < 4 \Rightarrow 0 \leq x^2 < 25$

Generalized Method of Intervals

Let $F(x) = (x - a_1)^{k_1}(x - a_2)^{k_2} \ldots.(x - a_{n-1})^{k_{n-1}}(x - a_n)^{k_n}$. Here, $k_1, k_2, \ldots k_n \in Z$ and $a_1, a_2, \ldots., a_n$ are fixed real number satisfying the condition

$a_1 < a_2 < a_3 < \ldots. < a_{n-1} < a_n$

* For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm :
  We mark the numbers $a_1, a_2, \ldots, a_n$ on the number axis and put sign in the interval on the right of the largest of these numbers, i.e., on the right of a_n.
* Then we put plus sign in the interval on the left of a_n if k_n is an even number and minus sign if k_n is an odd number. In the next interval, we put a sign according to the following rule :
  - When passing through the point a_{n-1}, the polynomial $F(x)$ changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.
* Thus, we consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all intervals in which we put plus sign and the solution of the inequality $F(x) < 0$ is the union of all intervals in which we put minus sign.

Frequently Used Inequalities

1. $(x - a)(x - b) < 0 \Rightarrow x \in (a, b)$ where $a < b$

2. $(x - a)(x - b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a < b$

3. $x^2 \leq a^2 \Rightarrow x \in [-a, a]$

4. $x^2 \geq a^2 \Rightarrow x \in (-\infty, a] \cup [a, \infty]$

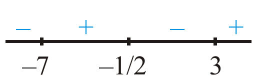
5. $ax^2 + bx + c < 0, (a > 0) \Rightarrow x \in (\alpha, \beta)$, where α, β $(\alpha < \beta)$ are the roots of the equation $ax^2 + bx + c = 0$

6. $ax^2 + bx + c > 0, (a > 0) \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where α, β $(\alpha < \beta)$ are the roots of the equation $ax^2 + bx + c = 0$

Ex. Solve $(2x + 1)(x - 3)(x + 7) < 0$

Sol. $(2x + 1)(x - 3)(x + 7) < 0$

The sign scheme of $(2x + 1)(x - 3)(x + 7)$ is as follows :



Hence, the solution is $(-\infty, -7) \cup (-1/2, 3)$.

Ex. Solve $\frac{2}{x} < 3$

Sol. $\frac{2}{x} < 3$

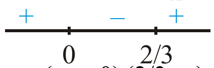
or $\frac{2}{x} - 3 < 0$ (We cannot cross multiply with x, as x can be negative or positive)

or $\frac{2-3x}{x} < 0$

or $\frac{3x-2}{x} > 0$

or $\frac{(x-2/3)}{x} > 0$

The sign scheme of $\frac{(x-2/3)}{x}$ is as follows :



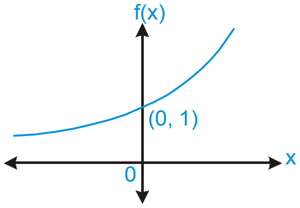
$\therefore$ $x \in (-\infty, 0)\,(2/3, \infty)$

Exponential Function

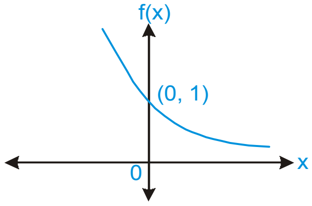
A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \neq 1, x \in R)$ is called an exponential function. Graph of exponential function can be as follows :

* Domain : R
* Range : $(0, \infty)$
* Nature : Non-periodic, one-one, neither odd nor even
* Monotonically increasing, wher $a > 0$
* Monotonically decreasing, when $0 < a < 1$

Case - I
For a > 1



Case - II
For 0 < a < 1



Definition of Logarithm

If a is a positive real number, other than 1 and x is a rational number such that $a^x = N$, then we say that logarithm of N to base a is x or x is the logarithm of N to base a, written as $\log_a N = x$,

Thus, $a^x = N \Leftrightarrow \log_a N = x$.

* Domain : $(0, \infty)$
* Range : $(-\infty, \infty)$
* Period : Non-periodic
* Nature : Neither odd nor even

Limitations of Logarithm: $\log_a N$ is defined only when

(i) $N > 0$ **(ii)** $a > 0$ **(iii)** $a \neq 1$

(i) For a given value of N, $\log_a N$ will give us a unique value.
(ii) Logarithm of zero does not exist.
(iii) Logarithm of negative reals are not defined in the system of real numbers.

Ex. Find the values of each of the following

(i) $\log_9 81$ **(ii)** $\log_{10} 0.0001$

Sol. (i) Let $\log_9 81 = x$. Then

$\log_9 81 = x$

$\Rightarrow \quad 9^x = 81$

$\Rightarrow \quad 9^x = 9^2 \quad \Rightarrow \quad x = 2$

(ii) Let $\log_{10} 0.0001 = x$.

Then, $\log_{10} 0.0001 = x$

$\Rightarrow \quad 10^x = 0.0001$

$\Rightarrow \quad 10^x = \dfrac{1}{10000}$

Logarithm of A Number

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$.

Hence : $\log_a N = x \Leftrightarrow a^x = N$, $a > 0$, $a \neq 1$ & $N > 0$

If $a = 10$, then we write $\log b$ rather than $\log_{10} b$.

If $a = e$, we write $\ln b$ rather than $\log_e b$.

The existence and uniqueness of the number $\log_a N$ follows from the properties of an exponential functions.

From the definition of the logarithm of the number N to the base 'a' , we have an identity : $a^{\log_a N} = N$, $a > 0$, $a \neq 1$ & $N > 0$

This is known as the Fundamental Logarithmic Identity.

$\log_a 1 = 0 \, (a > 0 \, , \, a \neq 1)$

$\log_a a = 1 \, (a > 0 \, , \, a \neq 1)$ and $\log_{1/a} a = -1 \, (a > 0 \, , \, a \neq 1)$

$\log_{10} 2 = 0.3010, \log_{10} 3 = 0.4771, \ln 2 = 0.693, \ln 10 = 2.3030$

Ex. If $\log_3 y = x$ and $\log_2 z = x$, find $72x$ in terms of y and z.

Sol. We have, $\log_3 y = x$ and $\log_2 z = x$

$\Rightarrow \quad y = 3^x$ and $z = 2^x$

Now, $72^x = (2^3 \times 3^2)^x$

$= (2^3)^x \times (3^2)^x \quad \Rightarrow \quad 2^{3x} . 3^{2x}$

$= (2^x)^3(3^x)^2$

$= (z)^3(y)^2$

$= y^2z^3.$

Ex. If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\dfrac{p^4q^4}{100} = 100^{2a-1}$

Sol. $\log_5 p = a$

$\Rightarrow \quad p = 5^a \qquad \log_2 q = a \quad \Rightarrow \quad q = 2^a$

$$\Rightarrow \quad \frac{p^4q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1}$$

System of Logarithm

There are two systems of logarithm which are generally used.

I. Common Logarithms : In this system the base is always taken as 10.

II. Natural Logarithm : In this system the base of the logarithm is taken as e, where e is an irrational number lying between 2 and 3.

Fundamental Laws of Logarithm

Let M & N are arbitrary positive numbers, $a > 0, a \neq 1, b > 0, b \neq 1$ and α is any real number then ;

(i) $\log_a (M . N) = \log_a M + \log_a N$

(ii) $\log_a (M/N) = \log_a M - \log_a N$

(iii) $\log_a M^b = b . \log_a M$

(iv) $\log_b M = \dfrac{\log_a M}{\log_a b}$ (base changing theorem)

(v) $\log_{a^b} m = \dfrac{1}{b}\log_a m$

(vi) $a^{\log_b c} = c^{\log_b a}$

(vii) $\log_{x^a} y^b = \dfrac{b}{a}\log_x y$

(viii) $\log_a x > \log_a y \Rightarrow \begin{cases} x > y, & \text{if } a > 1 \\ x < y, & \text{if } 0 < a < 1 \end{cases}$

(ix) If $\log_a x > y \Rightarrow \begin{cases} x > a^y, & \text{if } a > 1 \\ x < a^y, & \text{if } 0 < a < 1 \end{cases}$

(i) $\log_b a . \log_a b = 1 \Leftrightarrow \log_b a = 1/\log_a b.$

(ii) $\log_b a . \log_c b . \log_a c = 1$

(iii) $\log_y x . \log_z y . \log_a z = \log_a x.$

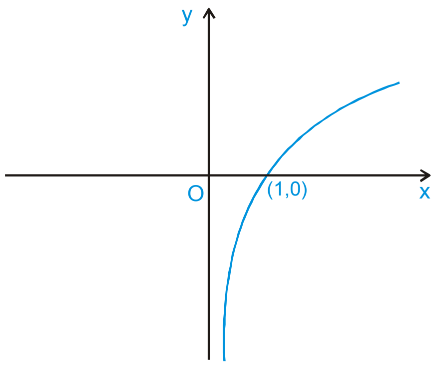
(iv) $e^{\ln a^x} = a^x$

(iv) $\log_e a = 2.303 \log_{10} a$

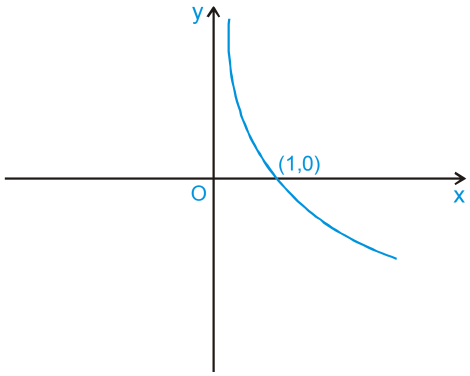
(v) $\log_{10} a = 0.434 \log_e a$

Graph of Logarithmic Function

Graph of $y = \log_a x$



When a > 1



When 0 < a < 1

(i) If base of logarithm is greater than 1 then logarithm of greater number is greater. i.e. $\log_2 8 = 3$, $\log_2 4 = 2$ etc. and if base of logarithm is between 0 and 1 then logarithm of greater number is smaller. i.e. $\log_{1/2} 8 = -3$, $\log_{1/2} 4 = -2$ etc.

(ii) It must be noted that whenever the number and the base are on the same side of unity then logarithm of that number to that base is positive, however if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

e.g. $\log_{10} \sqrt[3]{10} = \frac{1}{3}$; $\log_{\sqrt{7}} 49 = 4$; $\log_{\frac{1}{2}}\left(\frac{1}{8}\right) = 3$; $\log_2\left(\frac{1}{32}\right) = -5$; $\log_{10}(0.001) = -3$

(iii) $x + \frac{1}{x} \geq 2$ if x is positive real number and $x + \frac{1}{x} \leq -2$ if x is negative real number

(iv) $n \geq 2,\ n \in N$ $\sqrt[n]{a} = a^{1/n}$ $\Rightarrow$ n^{th} root of 'a' ('a' is a non negative number)

Logarithmic Equations

In this section, we shall learn about the methods of solving logarithmic equations which are most often used. In solving logarithmic equations, we first find set of values of variable(s) for which the given equation is meaningful and then it is solved by using laws of logarithms studied in the previous section.

The equality $\log_a x = \log_a y$ is possible if and only if x = y i.e.

$\log a\ x = \log a\ y \Leftrightarrow x = y$

Always check validity of given equation, $(x > 0, y > 0, a > 0, a \neq 1)$

Ex. $\log_3(x+1) + \log_3(x+3) = 1$

Sol. Domain : $x + 1 > 0$

$x + 3 > 0$

$\log_3(x+1)(x+3) = 1$

$\log_3(x^2 + 4x + 3) = 1$

$x^2 + 4x + 3 = 3 \quad \Rightarrow \quad x = 0, -4$ (not in domain)

$x = 0$

Ex. Find the values of x in each of the following :

(i) $\log_2 x = 3$ **(ii)** $\log_{81} x = \frac{3}{2}$

Sol. **(i)** $\log_2 x = 3 \quad \Rightarrow \quad x = (81)^{3/2}$

(ii) $\log_{81} x = \frac{3}{2} \quad \Rightarrow \quad x = (3^4)^{3/2}$

$\Rightarrow \quad x = 3^6 = 729$

Ex. $\log_x(2 - x) = 2$

Sol. $\log_x(2 - x) = 2$

$\Rightarrow \quad 2 - x = x^2$

$\Rightarrow \quad x^2 + x - 2 = 0$

$x = -2, 1$

But not satisfied the domain therefore no solution.

Standard Form of a Number

Standard form is a way of writing down very large or very small number easily. We can express a positive number in decimal form as the product of an integral power of 10 and a number between 1 to 10. That is any quality number n in decimal form is written as $\Rightarrow n = m \times 10^p$ where p is an integer and $1 \le m < 10$.

Ex. Express 35. 7 in the standard form of decimal.

Sol. Since 35.7 lies between 10 and 100. So, we write

$$35.7 = \frac{35.7}{10} \times 10 = 3.57 \times 10^1.$$

Characteristic and Mantissa

Let n be a positive real number and let $m \times 10^p$ be the standard of n. Then, $n = m \times 10^p$ where p is an integer and m is a real number between 1 and 10, i.e,. $1 \le m < 10$

$$\Rightarrow \quad \log n = \log(m \times 10^p)$$

$$= \log m + \log 10^p$$

$$= \log m + p \log 10$$

$$= p + \log m \qquad [\because \log 10 = \log_{10} 10 = 1]$$

Here p is an integer and $1 \leq m < 10$

now, $1 \leq m < 10$

$\Rightarrow$ $\log 1 \leq \log m < \log 10$

$\Rightarrow$ $0 \leq \log m < 1$

Thus, the logarithm of positive real number n consists of two parts :

(i) The integral part p, which is an integer : positive, negative or zero, and

(ii) The decimal part log m, which is a real number between 0 and 1.

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction

The integer part is called characteristic and the fractional part is called mantissa. When the value of log n is given, then to find digits of 'n' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $n \geq 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < n < 1$).

(i) The mantissa part of logarithm of a number is always positive ($0 \leq m < 1$)

(ii) If the characteristic of $\log_{10} N$ be n, then the number of digits in N is $(n + 1)$

(iii) If the characteristic of $\log_{10} N$ be $(-n)$, then there exist $(n - 1)$ zeros after decimal in N.

Ex. Write the characteristic of the logarithms of 2315.4 by using standard form :

Sol. $2315.4 = 2.3154 \times 10^3$ (Standard form)

characteristic = 3

Ex. Write the characteristic of the logarithms of 2315.4 without using standard form :

Sol. 2315.4

Number of digit to the left of the Decimal point = 4

So characteristic = 3

Ex. Find the total number of digits in the number 6^{100} .

(Given $\log_{10} 2 = 0.3010$; $\log_{10} 3 = 0.4771$)

Sol. $N = 6^{100}$

$\log_{10} N = 100 \log_{10} 6 = 10\,(0.3010 + .4771) = 77.8$

Characteristic $= [\log_{10} N]$

No. of digits = 77 + 1 = 78

Antilogarithm

The positive real number 'n' is called the antilogarithm of a number 'm' if log n = m

Thus, $\log n = m \Leftrightarrow n = \text{antilog } m$

Ex. **(i)** $\log 100 = 2 \Leftrightarrow \text{antilog } 2 = 100$

(ii) $\log 431.5 = 2.6350 \Leftrightarrow \text{antilog } (2.6350) = 431.5$

Some Important Identities

(1) $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$

(2) $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$

(3) $a^2 - b^2 = (a + b)(a - b)$

(4) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(5) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(6) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$

(7) $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$

(8) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

(9) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(10) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$$= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

(11) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$

(12) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

Frequently Used Inequalities

1. $(x - a)(x - b) < 0 \Rightarrow x \in (a, b)$ where $a < b$

2. $(x - a)(x - b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a < b$

3. $x^2 \le a^2 \Rightarrow x \in [-a, a]$

4. $x^2 \ge a^2 \Rightarrow x \in (-\infty, a] \cup [a, \infty]$

5. $ax^2 + bx + c < 0, (a > 0) \Rightarrow x \in (\alpha, \beta)$, where α, β $(\alpha < \beta)$ are the roots of the equation $ax^2 + bx + c = 0$

6. $ax^2 + bx + c > 0, (a > 0) \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where α, β $(\alpha < \beta)$ are the roots of the equation $ax^2 + bx + c = 0$

logarithm of a number

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$.

(a) $\log_a N = x$, read as log of N to the base a $\Leftrightarrow a^x = N$

If $a = 10$ then we write log N or $\log_{10} N$ and if $a = e$ we write

ln N or $\log_e N$ (Natural log)

(b) Necessary conditions : $N > 0$; $a > 0$; $a \neq 1$

(c) $\log_a 1 = 0$

(d) $\log_a a = 1$

(e) $\log_{1/a} a = -1$

(f) $\log_a(x.y) = \log_a x + \log_a y \; ; x, y > 0$

(g) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y \; ; x, y > 0$

(h) $\log_a x^p = p\log_a x \; ; x > 0$

(i) $\log_{a^q} x = \frac{1}{q}\log_a x \; ; x > 0$

(j) $\log_a x = \frac{1}{\log_x a} \; ; x > 0, x \neq 1$

(k) $\log_a x = \log_b x / \log_b a \; ; x > 0, a, b > 0, b \neq 1, a \neq 1$

(l) $\log_a b . \log_b c . \log_c d = \log_a d \; ; a, b, c, d > 0, \neq 1$

(m) $a^{\log_a x} = x ; a > 0, a \neq 1$

(n) $a^{\log_b c} = c^{\log_b a} \; ; a, b, c > 0 ; b \neq 1$

(o) $\log_a x < \log_a y \quad \Leftrightarrow \quad \begin{bmatrix} x < y & \text{if } a > 1 \\ x < y \text{ if} & 0 < a < 1 \end{bmatrix}$

(p) $\log_a x = \log_a y \quad \Rightarrow \quad x = y ; x, y > 0 ; a > 0, a \neq 1$

(q) $e^{\ln a^x} = a^x$

(r) $\log_{10} 2 = 0.03010 ; \log_{10} 3 = 0.4771 ; \ell n2 = 0.693, \ell n10 = 2.303$

(s) If $a > 1$ then $\log_a x < p \quad \Rightarrow \quad 0 < x < a^p$

(t) If $a > 1$ then $\log_a x > p \quad \Rightarrow \quad x > a^p$

(u) If $0 < a < 1$ then $\log_a x < p \quad \Rightarrow \quad x > a^p$

(v) If $0 < a < 1$ then $\log_a x > p \quad \Rightarrow \quad 0 < x < a^p$

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